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STRESS SINGULARITIES IN LAMINATED COMPOSITE WEDGES

I. O. OJIKUTU,† R. D. LOW‡ and R. A. SCOTT
Department of Mechanical Engineering and Applied Mechanics, University of Michigan, Ann Arbor, MI 48109, U.S.A.

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Abstract—This work is concerned with stress singularities at the apex of both symmetric and antisymmetric N-layered composite laminate wedges, using classical lamination theory. The symmetric bending case is governed by a single differential equation, while the antisymmetric case, in which bending and in-plane extension do not occur independently, involves three coupled differential equations. In each case, the governing differential equation(s) have non-constant coefficients which depend on the polar coordinate θ. These do not, in general, have closed-form solutions and numerical techniques must be employed. Finite difference schemes are used here. Results are presented for symmetric and antisymmetric configurations of graphite/epoxy (T-300/5208) angle-ply wedges.

1. INTRODUCTION

Because of their attractive strength—and stiffness-to-weight ratios, many aspects of composite media have recently been the focus of intense investigations. In pioneering works on stress singularities, Williams[1, 2] investigated the bending and in-plane extension of homogeneous, isotropic, elastic sector plates, subjected to various homogeneous boundary conditions. Chapkis and Williams[3] and Delale *et al.*[4] extended this type of analysis to polarly orthotropic media. Later, Dempsey and Sinclair[5], considering a linear, homogeneous elastic wedge, investigated the conditions necessary for the existence of a "Williams-type" singularity. Their analysis showed that for certain non-homogeneous boundary conditions, logarithmic singularities arise in addition to the Williams type. A certain pathological case was subsequently removed by Dempsey[6]. Such studies were further extended by Dempsey and Sinclair[7] to a bi-material wedge, thereby amplifying original work of Hein and Erdogan[8] in this area. In[9], Ting and Chou studied the wedge problem using linear anisotropic elasticity. They presented general methods, but no specific results were given. In view of this and the fact that it is not clear how their technique could be generalized to N-layered structures such as the one at hand, it was not pursued further.

The current work is concerned with one facet of the behavior of layered, sector plates subject to various homogeneous boundary conditions. Specifically, the behavior of the stress field in the vicinity of the apex of a wedge-shaped layered plate is sought .... the individual layers consisting of unidirectional, fiber reinforced laminae. In contrast to previous work involving polarly orthotropic media, the more realistic case of angle-ply laminates leads to considerable complications. A system of ordinary differential equations with non-constant coefficients ultimately arises for which no closed form solution exists. A numerical approach using finite differences is employed. Results are presented for symmetric and antisymmetric configurations of graphite epoxy (T-300/5208) angle-ply wedges.

2. THEORETICAL DEVELOPMENTS

The laminate treated is a sector plate made of N perfectly bonded, fiber reinforced layers with alternating ply angles. Classical plate lamination theory is employed in the sense that normals to the laminate's midsurface are assumed to remain normal. This "plane sections remain plane" assumption is felt to be adequate for the very thin layers that usually arise in practice. It is interesting to note that Ojikutu in[10] attempted to assess the accuracy of this using a theory which allowed each layer to have different rotations.

†Present address: Lecturer, Mathematics Department, University of Sokoto, Nigeria.
‡Deceased.

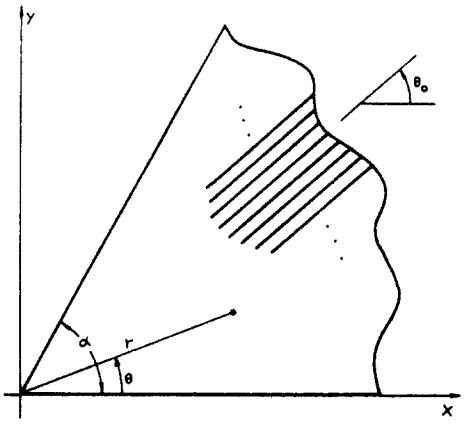

Fig. 1. Geometry.

However the attempt failed in that the resulting theory did not admit "Williams-type" solutions. The displacement components are (see Fig. 1).

$$u = u_0 - z\frac{\partial w_0}{\partial r}, \tag{1}$$

$$v = v_0 - \frac{z}{r}\frac{\partial w_0}{\partial \theta}, \tag{2}$$

$$w = w_0, \tag{3}$$

where $u_0(r, \theta)$, $v_0(r, \theta)$, and $w_0(r, \theta)$ are the displacement components of a point on the midsurface in the radial, tangential, and transverse directions, respectively. The pertinent strain displacement relations are then given by

$$\epsilon_r = \frac{\partial u_0}{\partial r} - z\frac{\partial^2 w_0}{\partial r^2}, \tag{4}$$

$$\epsilon_\theta = \frac{1}{r}\frac{\partial v_0}{\partial \theta} + \frac{u_0}{r} - z\left(\frac{1}{r}\frac{\partial w_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_0}{\partial \theta^2}\right). \tag{5}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u_0}{\partial \theta} + \frac{\partial v_0}{\partial r} - \frac{v_0}{r} + 2z\left(\frac{1}{r^2}\frac{\partial w_0}{\partial \theta} - \frac{1}{r}\frac{\partial^2 w_0}{\partial r \partial \theta}\right). \tag{6}$$

The stress-strain relations in polar coordinates are needed. In[10] it is shown that

$$\begin{Bmatrix} \sigma_r^{(k)} \\ \sigma_\theta^{(k)} \\ \tau_{r\theta}^{(k)} \end{Bmatrix} = \begin{pmatrix} \hat{Q}_{11}^{(k)} & \hat{Q}_{12}^{(k)} & \hat{Q}_{16}^{(k)} \\ \hat{Q}_{12}^{(k)} & \hat{Q}_{22}^{(k)} & \hat{Q}_{26}^{(k)} \\ \hat{Q}_{16}^{(k)} & \hat{Q}_{26}^{(k)} & \hat{Q}_{66}^{(k)} \end{pmatrix} \begin{Bmatrix} \epsilon_r^{(k)} \\ \epsilon_\theta^{(k)} \\ \gamma_{r\theta}^{(k)} \end{Bmatrix}, \tag{7}$$

where

$$\hat{Q}_{11}^{(k)} = \bar{Q}_{11}^{(k)}c^4 + 4\bar{Q}_{16}^{(k)}c^3 s + (2\bar{Q}_{12}^{(k)} + 4\bar{Q}_{66}^{(k)})c^2 s^2 + 4\bar{Q}_{26}^{(k)}cs^3 + \bar{Q}_{22}^{(k)}s^4, \tag{8}$$

$$\hat{Q}_{12}^{(k)} = \bar{Q}_{12}^{(k)}c^4 - (2\bar{Q}_{16}^{(k)} - 2\bar{Q}_{26}^{(k)})c^3 s + (\bar{Q}_{11}^{(k)} + \bar{Q}_{22}^{(k)} - 4\bar{Q}_{66}^{(k)})c^2 s^2 + (2\bar{Q}_{16}^{(k)} - 2\bar{Q}_{26}^{(k)})cs^3 + \bar{Q}_{12}^{(k)}s^4, \tag{9}$$

$$\hat{Q}_{16}^{(k)} = \bar{Q}_{16}^{(k)}c^4 - (\bar{Q}_{11}^{(k)} - \bar{Q}_{12}^{(k)} - 2\bar{Q}_{66}^{(k)})c^3 s - (3\bar{Q}_{16}^{(k)} - 3\bar{Q}_{26}^{(k)})c^2 s^2 - [\bar{Q}_{12}^{(k)} - \bar{Q}_{22}^{(k)} + 2\bar{Q}_{66}^{(k)}]cs^3 - \bar{Q}_{26}^{(k)}s^4, \tag{10}$$

$$\bar{Q}_{22}^{(k)} = \bar{Q}_{22}^{(k)}c^4 - 4\bar{Q}_{26}^{(k)}c^3s + [2\bar{Q}_{12}^{(k)} + 4\bar{Q}_{66}^{(k)}]c^2s^2 - 4\bar{Q}_{16}^{(k)}cs^3 + \bar{Q}_{11}^{(k)}s^4, \tag{11}$$

$$\bar{Q}_{26}^{(k)} = \bar{Q}_{26}^{(k)}c^4 - [\bar{Q}_{12}^{(k)} - \bar{Q}_{22}^{(k)} + 2\bar{Q}_{66}^{(k)}]c^3s + [3\bar{Q}_{16}^{(k)} - 3\bar{Q}_{26}^{(k)}]c^2s^2 - [\bar{Q}_{11}^{(k)} - \bar{Q}_{12}^{(k)} - 2\bar{Q}_{66}^{(k)}]cs^3 - \bar{Q}_{16}^{(k)}s^4, \tag{12}$$

$$\bar{Q}_{66}^{(k)} = \bar{Q}_{66}^{(k)}c^4 - [2\bar{Q}_{16}^{(k)} - 2\bar{Q}_{26}^{(k)}]c^3s + [\bar{Q}_{11}^{(k)} + \bar{Q}_{22}^{(k)} - 2\bar{Q}_{12}^{(k)} - 2\bar{Q}_{66}^{(k)}]c^2s^2 + 2[\bar{Q}_{16}^{(k)} - 2\bar{Q}_{26}^{(k)}]cs^3 + \bar{Q}_{66}^{(k)}s^4. \tag{13}$$

Here c and s stands for $\cos\theta$ and $\sin\theta$, respectively. The transformed stiffness $\bar{Q}_{ij}$ are related to the reduced stiffnesses Q_{ij} via the fiber angle θ_0. A typical one of these relations is

$$\bar{Q}_{22}^{(k)} = Q_{11}\sin^4\theta_0 + 2(Q_{12} + 2Q_{66})\sin^2\theta_0\cos^2\theta_0 + Q_{22}\cos^4\theta_0. \tag{14}$$

The others are similar and may be found in Jones ([11], p. 51).

The principle of virtual work states that, for zero body forces,

$$\int_S T_i\delta u_i\,dS = \int_V \sigma_{ij}\delta\epsilon_{ij}\,dV. \tag{15}$$

Since the plate consists of N layers, the kth being bounded by: $z_{k-1} \le z \le z_k$, (15) leads to, on using (4)–(7)

$$\int_0^\alpha\int_{r_i}^{r_0}\left\{\sum_{k=1}^{N}\int_{z_{k-1}}^{z_k} z\left[\sigma_r^{(k)}\frac{\partial^2}{\partial r^2}(\delta w_0) + \frac{\sigma_\theta^{(k)}}{r^2}\frac{\partial^2}{\partial\theta^2}(\delta w_0) + \frac{2\tau_{r\theta}^{(k)}}{r}\frac{\partial^2}{\partial r\partial\theta}(\delta w_0) + \frac{\sigma_\theta^{(k)}}{r}\frac{\partial}{\partial r}(\delta w_0) - \frac{2\tau_{r\theta}^{(k)}}{r^2}\frac{\partial}{\partial\theta}(\delta w_0)\right]dz\right\} r\,dr\,d\theta$$

$$-\int_0^\alpha\int_{r_i}^{r_0}\left\{\sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}\left[\sigma_r^{(k)}\frac{\partial}{\partial r}(\delta u_0) + \frac{\sigma_\theta^{(k)}}{r}\frac{\partial}{\partial\theta}(\delta v_0) + \frac{\sigma_\theta^{(k)}}{r}\delta u_0 + \frac{\tau_{r\theta}^{(k)}}{r}\frac{\partial}{\partial\theta}(\delta u_0) + \tau_{r\theta}^{(k)}\frac{\partial}{\partial r}(\delta v_0) - \frac{\tau_{r\theta}^{(k)}}{r}\delta v_0\right]dz\right\} r\,dr\,d\theta + \int_0^\alpha\int_{r_i}^{r_0} q\,\delta w_0\,r\,dr\,d\theta + \delta W_b = 0, \tag{16}$$

where α is the sector angle, r_i, r_0 are the inner and outer radii, respectively, of the plate (r_i is ultimately set = 0), $q(r,\theta)$ is the lateral loading, and δW_b is the virtual work done by the boundary tractions. In the applications to be considered here, the boundary conditions are such that δW_b is zero. Other types of boundry conditions are treated in [10].

Stress resultants are introduced by using the same terminology as in Jones [11]:

$$(N_r, N_\theta, N_{r\theta}) = \sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}(\sigma_r^{(k)}, \sigma_\theta^{(k)}, \tau_{r\theta}^{(k)})\,dz, \tag{17}$$

$$(M_r, M_\theta, M_{r\theta}) = \sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}(\sigma_r^{(k)}, \sigma_\theta^{(k)}, \tau_{r\theta}^{(k)})z\,dz. \tag{18}$$

Then (16) gives, after several integrations by parts:

$$\int_0^\alpha\int_{r_i}^{r_0}\left[r\frac{\partial^2 M_r}{\partial r^2} + 2\frac{\partial^2 M_{r\theta}}{\partial r\partial\theta} + \frac{1}{r}\frac{\partial^2 M_\theta}{\partial\theta^2} + 2\frac{\partial M_r}{\partial r} + \frac{2}{r}\frac{\partial M_{r\theta}}{\partial\theta} - \frac{\partial M_\theta}{\partial r} + rq\right]\delta w_0\,dr\,d\theta$$

$$+\int_0^\alpha\int_{r_i}^{r_0}\left[r\frac{\partial N_r}{\partial r} + \frac{\partial N_{r\theta}}{\partial\theta} + N_r - N_\theta\right]\delta u_0\,dr\,d\theta + \int_0^\alpha\int_{r_i}^{r_0}\left[r\frac{\partial N_{r\theta}}{\partial r} + \frac{\partial N_\theta}{\partial\theta} + 2N_{r\theta}\right]\delta v_0\,dr\,d\theta$$

$$+\int_0^\alpha\left[rN_r\delta u_0 + rN_{r\theta}\delta v_0 + rV_r\delta w_0 - rM_r\delta\left(\frac{\partial w_0}{\partial r}\right)\right]_{r_i}^{r_0}d\theta$$

$$+\int_{r_i}^{r_0}\left[N_{r\theta}\delta u_0 + N_\theta\delta v_0 + V_\theta\delta w_0 - M_\theta\delta\left(\frac{1}{r}\frac{\partial w_0}{\partial\theta}\right)\right]_0^\alpha dr, \tag{19}$$

where

$$V_r = \frac{\partial M_r}{\partial r} + \frac{2}{r}\frac{\partial M_{r\theta}}{\partial \theta} + \frac{M_r - M_\theta}{r},$$
$$V_\theta = 2\frac{\partial M_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial M_\theta}{\partial \theta} + \frac{2}{r}M_{r\theta}. \tag{20}$$

Hence we have the differential equations:

$$\frac{\partial N_r}{\partial r} + \frac{1}{r}\frac{\partial N_{r\theta}}{\partial \theta} + \frac{N_r - N_\theta}{r} = 0, \tag{21}$$

$$\frac{\partial N_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial N_\theta}{\partial \theta} + \frac{2N_{r\theta}}{r} = 0, \tag{22}$$

$$\frac{\partial^2 M_r}{\partial r^2} + \frac{2}{r}\frac{\partial^2 M_{r\theta}}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial^2 M_\theta}{\partial \theta^2} + \frac{2}{r}\frac{\partial M_r}{\partial r} + \frac{2}{r^2}\frac{\partial M_{r\theta}}{\partial \theta} - \frac{1}{r}\frac{\partial M_\theta}{\partial r} + q = 0, \tag{23}$$

Appropriate homogeneous boundary conditions can be deduced from the boundary terms in (19). There are four types of simply-supported edges and four types of clamped edges (see Chia[12]). The only one pursued here is that simply supported case in which the support is free to move in the plane of the plate. In this case:

$$w_0 = 0, \quad N_\theta = 0, \quad N_{r\theta} = 0, \quad M_\theta = 0 \quad \text{on} \quad \theta = 0, \quad \alpha. \tag{24}$$

Using (4)–(7), (17) and (18), and (21)–(23), displacement equations can be obtained. In these, seeking "Williams-type" singularities, one sets

$$u_0 = r^\lambda F_1(\theta, \lambda), \quad v_0 = r^\lambda F_2(\theta, \lambda), \quad w_0 = r^{\lambda+1} F_3(\theta, \lambda). \tag{25}$$

One finally obtains

$$\begin{aligned} &C_3F_1'' + [C_2\lambda + C_5]F_1' + [C_1\lambda(\lambda-1) + C_4\lambda + C_6]F_1 + C_9F_2'' + [C_8\lambda + C_{11}]F_2' \\ &\quad + [C_7\lambda(\lambda-1) + C_{10}\lambda + C_{12}]F_2 + C_{16}F_3''' + [C_{15}(\lambda+1) + C_{19}]F_3'' \\ &\quad + [C_{14}(\lambda+1)\lambda + C_{18}(\lambda+1) + C_{21}]F_3' + [C_{13}(\lambda+1)\lambda(\lambda-1) + C_{17}(\lambda+1)\lambda \\ &\quad + C_{20}(\lambda+1)]F_3 = 0, \end{aligned} \tag{26}$$

$$\begin{aligned} &C_{24}F_1'' + [C_{23}\lambda + C_{26}]F_1' + [C_{22}\lambda(\lambda-1) + C_{25}\lambda + C_{27}]F_1 + C_{30}F_2'' + [C_{29}\lambda + C_{32}]F_2' \\ &\quad + [C_{28}\lambda(\lambda-1) + C_{31}\lambda + C_{33}]F_2 + C_{37}F_3''' + [C_{36}(\lambda+1) + C_{40}]F_3'' + [C_{35}(\lambda+1)\lambda \\ &\quad + C_{39}(\lambda+1) + C_{42}]F_3' + [C_{34}(\lambda+1)\lambda(\lambda-1) + C_{38}(\lambda+1)\lambda + C_{41}(\lambda+1)]F_3 = 0, \end{aligned} \tag{27}$$

$$\begin{aligned} &C_{47}F_3'''' + [C_{46}(\lambda+1) + C_{51}]F_3''' + [C_{45}(\lambda+1)\lambda + C_{50}(\lambda+1) + C_{54}]F_3'' + [C_{44}(\lambda+1)\lambda(\lambda-1) \\ &\quad + C_{49}(\lambda+1)\lambda + C_{53}(\lambda+1) + C_{56}]F_3' + [C_{43}(\lambda+1)\lambda(\lambda-1)(\lambda-2) \\ &\quad + C_{48}(\lambda+1)\lambda(\lambda-1) + C_{52}(\lambda+1)\lambda + C_{55}(\lambda+1)]F_3 + C_{60}F_1''' + [C_{59}\lambda + C_{63}]F_1'' \\ &\quad + [C_{58}\lambda(\lambda-1) + C_{62}\lambda + C_{65}]F_1' + [C_{57}\lambda(\lambda-1)(\lambda-2) \\ &\quad + C_{61}\lambda(\lambda-1) + C_{64}\lambda + C_{66}]F_1 + C_{70}F_2''' + [C_{69}\lambda + C_{73}]F_2'' + [C_{68}\lambda(\lambda-1) + C_{72}\lambda \\ &\quad + C_{75}]F_2' + [C_{67}\lambda(\lambda-1)(\lambda-2) + C_{71}\lambda(\lambda-1) + C_{74}\lambda + C_{76}]F_2 = 0. \end{aligned} \tag{28}$$

The $C_k(\theta)$ in (26)–(28) are given by

$$C_1 = \hat{A}_{11}, \quad C_2 = 2\hat{A}_{16}, \quad C_3 = \hat{A}_{66}, \quad C_4 = \hat{A}_{11} + \frac{d\hat{A}_{16}}{d\theta},$$

$$C_5 = \frac{d\hat{A}_{66}}{d\theta}, \quad C_6 = \frac{d\hat{A}_{26}}{d\theta} - \hat{A}_{22}, \quad C_7 = \hat{A}_{16},$$

$$C_8 = \hat{A}_{12} + \hat{A}_{66}, \quad C_9 = \hat{A}_{26}, \quad C_{10} = \frac{d\hat{A}_{66}}{d\theta} - \hat{A}_{26},$$

$$C_{11} = \frac{d\hat{A}_{26}}{d\theta} - \hat{A}_{22} - \hat{A}_{66}, \quad C_{12} = \hat{A}_{26} - \frac{d\hat{A}_{66}}{d\theta},$$

$$C_{13} = -\hat{B}_{11}, \quad C_{14} = -3\hat{B}_{16}, \quad C_{15} = -\hat{B}_{12} - 2\hat{B}_{66},$$

$$C_{16} = -\hat{B}_{26}, \quad C_{17} = -\hat{B}_{11} - \frac{d\hat{B}_{16}}{d\theta},$$

$$C_{18} = 2\hat{B}_{16} + \hat{B}_{26} - 2\frac{d\hat{B}_{66}}{d\theta}, \quad C_{19} = \hat{B}_{12} + \hat{B}_{22} + 2\hat{B}_{66} - \frac{d\hat{B}_{26}}{d\theta},$$

$$C_{20} = \hat{B}_{22} - \frac{d\hat{B}_{26}}{d\theta}, \quad C_{21} = -2\hat{B}_{16} - \hat{B}_{26} + 2\frac{d\hat{B}_{66}}{d\theta},$$

$$C_{22} = \hat{A}_{16}, \quad C_{23} = \hat{A}_{12} + \hat{A}_{66}, \quad C_{24} = \hat{A}_{26},$$

$$C_{25} = 2\hat{A}_{16} + \hat{A}_{26} + \frac{d\hat{A}_{12}}{d\theta}, \quad C_{26} = \hat{A}_{22} + \hat{A}_{66} + \frac{d\hat{A}_{26}}{d\theta},$$

$$C_{27} = \hat{A}_{26} + \frac{d\hat{A}_{22}}{d\theta}, \quad C_{28} = \hat{A}_{66}, \quad C_{29} = 2\hat{A}_{26},$$

$$C_{30} = \hat{A}_{22}, \quad C_{31} = \hat{A}_{66} + \frac{d\hat{A}_{26}}{d\theta}, \quad C_{32} = \frac{d\hat{A}_{22}}{d\theta},$$

$$C_{33} = -\hat{A}_{66} - \frac{d\hat{A}_{26}}{d\theta}, \quad C_{34} = -\hat{B}_{16}, \quad C_{35} = -\hat{B}_{12} - 2\hat{B}_{66},$$

$$C_{36} = -3\hat{B}_{26}, \quad C_{37} = -\hat{B}_{22}, \quad C_{38} = -2\hat{B}_{16} - \hat{B}_{26} - \frac{d\hat{B}_{12}}{d\theta},$$

$$C_{39} = -\hat{B}_{22} - 2\frac{d\hat{B}_{26}}{d\theta}, \quad C_{40} = 2\hat{B}_{26} - \frac{d\hat{B}_{22}}{d\theta},$$

$$C_{41} = -\hat{B}_{26} - \frac{d\hat{B}_{22}}{d\theta}, \quad C_{42} = 2\frac{d\hat{B}_{22}}{d\theta},$$

$$C_{43} = \hat{D}_{11}, \quad C_{44} = 4\hat{D}_{16}, \quad C_{45} = 2\hat{D}_{12} + 4\hat{D}_{66},$$

$$C_{46} = 4\hat{D}_{26}, \quad C_{47} = \hat{D}_{22}, \quad C_{48} = 2\hat{D}_{11} + 2\frac{d\hat{D}_{16}}{d\theta},$$

$$C_{49} = 2\frac{d\hat{D}_{12}}{d\theta} + 4\frac{d\hat{D}_{66}}{d\theta}, \quad C_{50} = -2\hat{D}_{12} - 4\hat{D}_{66} + 6\frac{d\hat{D}_{26}}{d\theta},$$

$$C_{51} = -4\hat{D}_{26} + 2\frac{d\hat{D}_{22}}{d\theta}, \quad C_{52} = -\hat{D}_{22} + 2\frac{d\hat{D}_{16}}{d\theta} + 2\frac{d\hat{D}_{26}}{d\theta} + \frac{d^2\hat{D}_{12}}{d\theta^2},$$

$$C_{53} = 4\hat{D}_{16} + 4\hat{D}_{26} + 2\frac{d\hat{D}_{22}}{d\theta} - 4\frac{d\hat{D}_{66}}{d\theta} + 2\frac{d^2\hat{D}_{26}}{d\theta^2},$$

$$C_{54} = 2\hat{D}_{12} + 2\hat{D}_{22} + 4\hat{D}_{66} - 6\frac{d\hat{D}_{26}}{d\theta} + \frac{d^2\hat{D}_{22}}{d\theta^2},$$

$$C_{55} = \hat{D}_{22} + \frac{d^2\hat{D}_{22}}{d\theta^2}, \quad C_{56} = -4\hat{D}_{16} - 4\hat{D}_{26} + 4\frac{d\hat{D}_{66}}{d\theta} - 2\frac{d^2\hat{D}_{26}}{d\theta^2},$$

$$C_{57} = -\hat{B}_{11}, \quad C_{58} = -3\hat{B}_{16}, \quad C_{59} = -\hat{B}_{12} - 2\hat{B}_{66},$$

$$C_{60} = -\hat{B}_{26}, \quad C_{61} = -2\hat{B}_{11} - 2\frac{d\hat{B}_{16}}{d\theta},$$

$$C_{62} = -\left[2\hat{B}_{16} + \hat{B}_{26} + 2\frac{d\hat{B}_{12}}{d\theta} + 2\frac{d\hat{B}_{66}}{d\theta}\right],$$

$$C_{63} = -\hat{B}_{22} - 2\frac{d\hat{B}_{26}}{d\theta}, \quad C_{64} = \hat{B}_{22} - 2\frac{d\hat{B}_{16}}{d\theta} - 2\frac{d\hat{B}_{26}}{d\theta} - \frac{d^2\hat{B}_{12}}{d\theta^2},$$

$$C_{65} = -\hat{B}_{26} - 2\frac{d\hat{B}_{22}}{d\theta} - \frac{d^2\hat{B}_{26}}{d\theta^2}, \quad C_{66} = -\hat{B}_{22} - \frac{d^2\hat{B}_{22}}{d\theta^2},$$

$$C_{67} = -\hat{B}_{16}, \quad C_{68} = -\hat{B}_{12} - 2\hat{B}_{66}, \quad C_{69} = -3\hat{B}_{26},$$

$$C_{70} = -\hat{B}_{22}, \quad C_{71} = -\hat{B}_{16} + \hat{B}_{26} - 2\frac{d\hat{B}_{66}}{d\theta},$$

$$C_{72} = \hat{B}_{22} - 4\frac{d\hat{B}_{26}}{d\theta}, \quad C_{73} = \hat{B}_{26} - 2\frac{d\hat{B}_{22}}{d\theta},$$

$$C_{74} = -\hat{B}_{26} - \frac{d^2\hat{B}_{26}}{d\theta^2}, \quad C_{75} = -\hat{B}_{22} + 2\frac{d\hat{B}_{26}}{d\theta} - \frac{d^2\hat{B}_{22}}{d\theta^2},$$

$$C_{76} = \hat{B}_{26} + \frac{d^2\hat{B}_{26}}{d\theta^2}, \tag{29}$$

In (29),

$$(\hat{A}_{ij}, \hat{B}_{ij}, \hat{D}_{ij}) = \sum_{k=1}^{N} \hat{Q}_{ij} \int_{z_{k-1}}^{z_k} (1, z, z^2)\, dz, \tag{30}$$

are, respectively, the extensional, coupling and bending stiffness.

Using (17), (18), (24) and (25), the boundary conditions for the problem are that on $\theta = 0$ and on $\theta = \alpha$, the following must hold:

$$F_3 = 0, \tag{31}$$

$$\hat{A}_{26}F_1' + (\lambda\hat{A}_{12} + \hat{A}_{22})F_1 + \hat{A}_{22}F_2' + (\lambda - 1)\hat{A}_{26}F_2 - \hat{B}_{22}F_3'' - 2\lambda\hat{B}_{26}F_3' - (\lambda + 1)(\lambda\hat{B}_{12} + \hat{B}_{22})F_3 = 0, \tag{32}$$

$$\hat{A}_{66}F_1' + (\lambda\hat{A}_{16} + \hat{A}_{26})F_1 + \hat{A}_{26}F_2' + (\lambda - 1)\hat{A}_{66}F_2 - \hat{B}_{26}F_3'' - 2\lambda\hat{B}_{66}F_3' - (\lambda + 1)(\lambda\hat{B}_{16} + \hat{B}_{26})F_3 = 0, \tag{33}$$

$$\hat{B}_{26}F + (\lambda\hat{B}_{12} + \hat{B}_{22})F_1 + \hat{B}_{22}F_2' + (\lambda - 1)\hat{B}_{26}F_2 - \hat{D}_{22}F_3'' - 2\lambda\hat{D}_{26}F_3' - (\lambda + 1)(\lambda\hat{D}_{12} + \hat{D}_{22})F_3 = 0. \tag{34}$$

Clearly, an analytic solution is not feasible so numerical techniques must be employed.

3. NUMERICAL PROCEDURE

The first case to be considered is the symmetric one. This occurs when N is odd and $z_{k-1} = z_{N-k+1}$, $k = 1, 2, \ldots, (N+1)/2$. It follows from (30) that in this case all the $\hat{B}_{ij}$ are zero. Then there is no coupling between bending and extension. Henceforth we consider bending only. Equations (28), (31) and (34) then reduce to

$$\begin{aligned} C_{47}F_3'''' + [C_{46}(\lambda+1) + C_{51}]F_3''' + [C_{45}(\lambda+1)\lambda + C_{50}(\lambda+1) + C_{54}]F_3'' \\ + [C_{44}(\lambda+1)\lambda(\lambda-1) + C_{49}(\lambda+1)\lambda + C_{53}(\lambda+1) + C_{56}]F_3' \\ + [C_{43}(\lambda+1)\lambda(\lambda-1)(\lambda-2) + C_{48}(\lambda+1)\lambda(\lambda-1) + C_{52}(\lambda+1)\lambda \\ + C_{55}(\lambda+1)]F_3 = 0 \end{aligned} \tag{35}$$

and

$$\left.\begin{aligned} &F_3 = 0, \\ &\hat{D}_{22}F_3'' + 2\hat{D}_{26}\lambda F_3' + (\lambda+1)(\hat{D}_{21}\lambda + \hat{D}_{22})F_3 = 0 \end{aligned}\right\} \theta = 0, \alpha$$

or, equivalently,

$$\left.\begin{aligned} &F_3 = 0 \\ &\hat{D}_{22}F_3'' + 2\hat{D}_{26}\lambda F_3' = 0 \end{aligned}\right\} \theta = 0,\ \alpha. \tag{36}$$

The first step in the numerical procedure is the use of the central difference scheme. The sector angle α is subdivided into $n+1$ parts with θ_0† $= 0$ and $\theta_{n+1} = \alpha$. Using the notation $f_k = F_3(\theta_k)$, $k = 1, 2, \ldots n$, the scheme gives for the n "inside" points:

$$N_1 f_{k-2} + N_2 f_{k-1} + N_3 f_k + N_4 f_{k+1} + N_5 f_{k+2} = 0, \tag{37}$$

in which

$$N_j = N_j(\theta_k, \lambda), \quad j = 1, \ldots, 5,$$

where

$$N_1(\theta_k, \lambda) = 2C_{47}(\theta_k) - [(\lambda+1)C_{46}(\theta_k) + C_{51}(\theta_k)]h, \tag{38}$$

$$\begin{aligned} N_2(\theta_k, \lambda) = &-8C_{47}(\theta_k) + 2[(\lambda+1)C_{46}(\theta_k) + C_{51}(\theta_k)]h \\ &+ 2[\lambda(\lambda+1)C_{45}(\theta_k) + (\lambda+1)C_{50}(\theta_k) + C_{54}(\theta_k)]h^2 - [\lambda(\lambda^2-1)C_{44}(\theta_k) \\ &+ \lambda(\lambda+1)C_{49}(\theta_k) + (\lambda+1)C_{53}(\theta_k) + C_{57}(\theta_k)]h^3, \end{aligned} \tag{39}$$

$$\begin{aligned} N_3(\theta_k, \lambda) = &\ 12C_{47}(\theta_k) - 4[\lambda(\lambda+1)C_{45}(\theta_k) + (\lambda+1)C_{50}(\theta_k) \\ &+ C_{59}(\theta_k)]h^2 + 2[\lambda(\lambda^2-1)(\lambda-2)C_{43}(\theta_k) + \lambda(\lambda^2-1)C_{48}(\theta_k) \\ &+ \lambda(\lambda+1)C_{52}(\theta_k) + (\lambda+1)C_{55}(\theta_k)]h^4, \end{aligned} \tag{40}$$

$$\begin{aligned} N_4(\theta_k, \lambda) = &-8C_{47}(\theta_k) - 2[(\lambda+1)C_{46}(\theta_k) + C_{51}(\theta_k)]h \\ &+ 2[\lambda(\lambda+1)C_{45}(\theta_k) + (\lambda+1)C_{50}(\theta_k) + C_{54}(\theta_k)]h^2 + [\lambda(\lambda^2-1)C_{44}(\theta_k) \\ &+ \lambda(\lambda+1)C_{49}(\theta_k) + (\lambda+1)C_{53}(\theta_k) + C_{56}(\theta_k)]h^3 \end{aligned} \tag{41}$$

$$N_5(\theta_k, \lambda) = 2C_{47}(\theta_k) + [(\lambda+1)C_{46}(\theta_k) + C_{51}(\theta_k)]h, \tag{42}$$

and h is the step size employed in the difference scheme.

†Not to be confused with the fiber angle.

Equation (37) is a system of n equations in $n+4$ unknowns $f_{-1}, f_0, f_1, \ldots, f_n, f_{n+1}, f_{n+2}$. The boundary conditions give

$$f_0 = 0, \quad f_{n+1} = 0,$$
$$(\hat{D}_{22} - \hat{D}_{26}\lambda h) f_{-1} + (\hat{D}_{22} + \hat{D}_{26}\lambda h) f_1 = 0, \quad (\hat{D}_{22} - \hat{D}_{26}\lambda h) f_n + (\hat{D}_{22} - \hat{D}_{26}\lambda h) f_{n+2} = 0. \tag{43}$$

For any n, (37) and (43) may be written

$$[A(\lambda)] \begin{Bmatrix} f_{-1} \\ f_1 \\ \vdots \\ f_n \\ f_{n+2} \end{Bmatrix} = 0, \tag{44}$$

where $[A(\lambda)]$ is an $n+2$ square matrix. For nontrivial solutions $\{f_k\}$,

$$\det(A(\lambda)) = 0. \tag{45}$$

To get some idea as to how large n should be in the computations, the bending of an isotropic plate with simply-supported edges at $\theta = 0$ and $\theta = a$ was first considered (see [10] for details). The degree of accuracy of the central difference scheme for $n = 6$, 12 and 30 is then tested.

The value of $\det(A(\lambda))$ is computed for each of a relatively widely spaced sequence of λ values, starting with $\lambda = 0$. For each λ, $\det(A(\lambda))$ is obtained by first performing an LU-decomposition on $A(\lambda)$ via the use of Scientific Subroutine program (NAAS:NAL's CDLUD subroutine (see [13])) and evaluation of the product of the diagonal elements. When a sign change in $\det(A(\lambda))$ is detected in an interval, finer subdivisions of this interval locates a "smallest" absolute value of $\det(A(\lambda))$. When $\det(A(\lambda)) \leqq 10^{-4}$, approx., λ will have been determined to 5 significant digits.

As shown in Table 1, values of $n = 6$, 12 and 30 give the desired root to within 15%, 4.5% and 0.8%, respectively for sector angles α up to 170°.

Using these procedures with $n = 24$,† results were obtained for a graphite/epoxy T-300/5208 angle-ply wedge of thickness 2 mm consisting of 7 layers of equal thickness. Shown in Figs. 2 and 3 are values of minimum Re λ for various sector angles α and fiber angles $[\theta_0/-\theta_0]$, $\theta_0 = 15°, 30°, \ldots, 90°$.

Inspection of these Figures shows that, as the fiber angle θ_0 increases, so does the sector angle α at which singular stresses begin to appear.... the ranges being:

θ_0:	15°	30°	45°	60°	75°	90°
α:	63°	73°	77°	83°	84°	85°.

The Figures also show that for sector angles in the range 90°–95°, the "strength" of the stress singularity is about the same for all fiber angles considered.... its λ-value being in the range 0.6–0.7.

Antisymmetric laminates will now be considered. In (26)–(28) let $F_1(\theta)$, $F_2(\theta)$, $F_3(\theta)$ have the functional values f_k, g_k, p_k in that order, at the n "inside points" θ_k, $k = 1, 2, \ldots, n$. Then one gets

$$(M_1) f_{k-1} + (M_2) f_k + (M_3) f_{k+1} + (M_4) g_{k-1} + (M_5) g_k + (M_6) g_{k-1} + (M_7) p_{k-2} + (M_8) p_{k-1} + (M_9) p_k + (M_{10}) p_{k+1} + (M_{11}) p_{k+2} = 0, \tag{46}$$

†Similar to the isotropic case, the values $n = 12$, 18, 24 and 30 were tested in computations for the largest sector angle. Since no significant deviation was noted between $n = 24$ and $n = 30$, it was more economical to obtain general results with the former.

Table 1. Minimum Re λ values for isotropic bending

α (°)	William's Solution	n = 6	n = 12	n = 30
20	8.00000	7.92502	7.97833	7.99760
40	3.50000	3.46023	3.48900	3.49861
60	2.00000	1.97274	1.99275	1.99880
80	1.25000	1.23116	1.24450	1.24905
90	1.00000	0.98326	0.99525	0.99917
100	0.80000	0.78493	0.79550	0.79923
110	0.63636	0.62266	0.63250	0.63566
120	0.50000	0.48744	0.49625	0.49937
130	0.38462	0.37302	0.38117	0.38403
140	0.28571	0.27495	0.28267	0.28517
150	0.20000	0.18995	0.19713	0.19948
160	0.12500	0.11558	0.12233	0.12453
170	0.05882	0.04996	0.05617	0.05838
180	0	0.01557	0.00245	0.00043

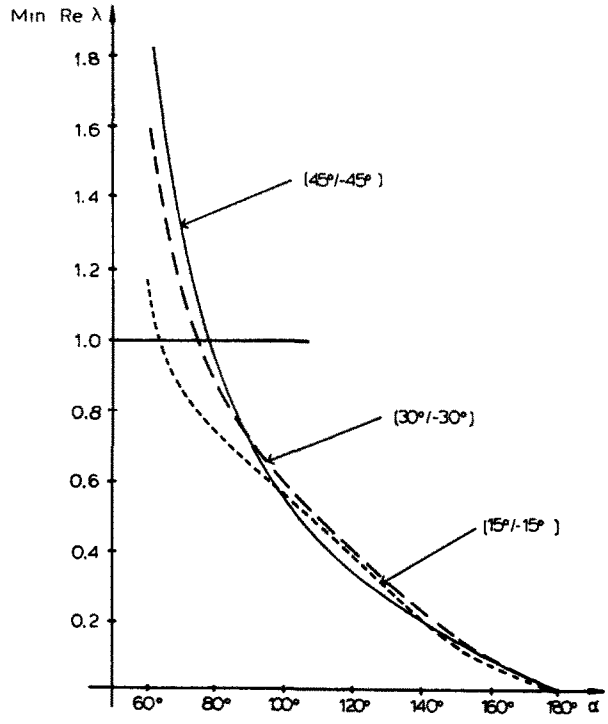

Fig. 2. Symmetric bending ($\theta_0 = 15°, 30°, 45°$): minimum Re λ vs sector α curves.

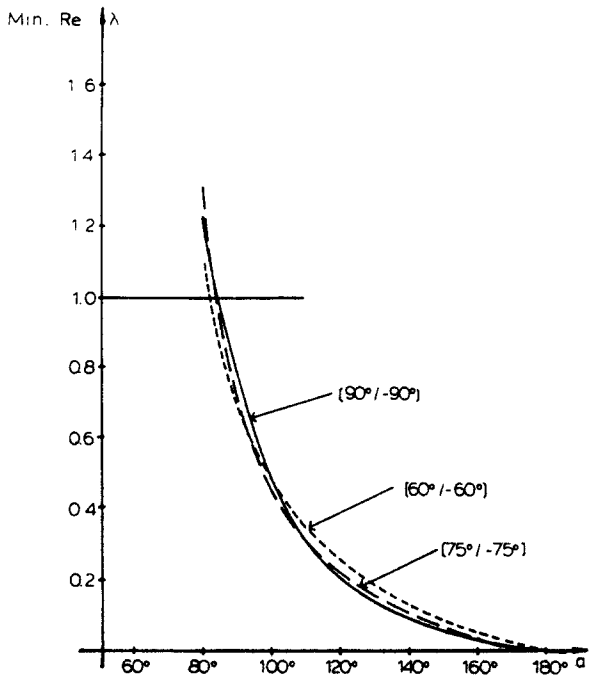

Fig. 3. Symmetric bending ($\theta_0 = 60°, 75°, 90°$): minimum Re λ vs sector angle α curves.

$$(M_{12})f_{k-1} + (M_{13})f_k + (M_{14})f_{k+1} + (M_{15})g_{k-1} + (M_{16})g_k + (M_{17})g_{k+1} + (M_{18})p_{k-2} + (M_{19})p_{k-1} + (M_{20})p_k + (M_{21})p_{k+1} + (M_{22})p_{k+2} = 0, \tag{47}$$

and

$$(M_{23})f_{k-2} + (M_{24})f_{k-1} + (M_{25})f_k + (M_{26})f_{k+1} + (M_{27})f_{k+2} + (M_{28})g_{k-2} + (M_{29})g_{k-1} + (M_{30})g_k + (M_{31})g_{k+1} + (M_{32})g_{k+2} + (M_{33})p_{k-2} + (M_{34})p_{k-1} + (M_{35})p_k + (M_{36})p_{k+1} + (M_{37})p_{k+2} = 0. \tag{48}$$

where

$$M_1 = 2C_3h - [C_2\lambda + C_5]h^2, \quad M_2 = -4C_3h + 2[C_1\lambda(\lambda - 1) + C_4\lambda + C_6]h^3$$
$$M_3 = 2C_3h + [C_2\lambda + C_5]h^2, \quad M_4 = 2C_9h - [C_8\lambda + C_{11}]h^2$$
$$M_5 = -4C_9h + 2[C_7\lambda(\lambda - 1) + C_{10}\lambda + C_{12}]h^3$$
$$M_6 = 2C_9h + [C_8\lambda + C_{11}]h^2, \quad M_7 = -C_{16}$$
$$M_8 = 2C_{16} + 2[C_{15}(\lambda + 1) + C_{19}]h - [C_{14}(\lambda + 1)\lambda + C_{18}(\lambda + 1) + C_{21}]h^2$$
$$M_9 = -4[C_{15}(\lambda + 1) + C_{19}]h + 2[C_{13}(\lambda + 1)\lambda(\lambda - 1) + C_{17}(\lambda + 1)\lambda + C_{20}(\lambda + 1)]h^3$$
$$M_{10} = -2C_{16} + 2[C_{15}(\lambda + 1) + C_{19}]h + [C_{14}(\lambda + 1)\lambda + C_{18}(\lambda + 1) + C_{21}]h^2$$
$$M_{11} = C_{16}, \quad M_{12} = 2C_{24}h - [C_{23}\lambda + C_{26}]h^2$$
$$M_{13} = -4C_{24}h + 2[C_{22}\lambda(\lambda - 1) + C_{25}\lambda + C_{27}]h^3$$
$$M_{14} = 2C_{24}h + [C_{23}\lambda + C_{26}]h^2, \quad M_{15} = 2C_{30}h - [C_{29}\lambda + C_{32}]h^2$$
$$M_{16} = -4C_{30}h + 2[C_{28}\lambda(\lambda - 1) + C_{31}\lambda + C_{33}]h^3$$

$$M_{17} = 2C_{30}h + [C_{29}\lambda + C_{32}]h^2, \quad M_{18} = -C_{37}$$

$$M_{19} = 2C_{37} + 2[C_{36}(\lambda+1) + C_{40}]h - [C_{35}(\lambda+1)\lambda + C_{39}(\lambda+1) + C_{42}]h^2$$

$$M_{20} = -4[C_{36}(\lambda+1) + C_{40}]h + 2[C_{34}(\lambda+1)\lambda(\lambda-1) + C_{38}(\lambda+1)\lambda + C_{41}(\lambda+1)]h^3$$

$$M_{21} = -2C_{37} + 2[C_{36}(\lambda+1) + C_{40}]h + [C_{35}(\lambda+1)\lambda + C_{39}(\lambda+1) + C_{42}]h^2, \quad M_{22} = C_{37}$$

$$M_{23} = -C_{60}h, \quad M_{24} = 2C_{60}h + 2[C_{59}\lambda + C_{63}]h^2 - [C_{58}\lambda(\lambda-1) + C_{62}\lambda + C_{65}]h^3$$

$$M_{25} = -4[C_{59}\lambda + C_{63}]h^2 + 2[C_{57}\lambda(\lambda-1)(\lambda-2) + C_{61}\lambda(\lambda-1) + C_{64}\lambda + C_{66}]h^4$$

$$M_{26} = -2C_{60}h + 2[C_{59}\lambda + C_{63}]h^2 + [C_{58}\lambda(\lambda-1) + C_{62}\lambda + C_{65}]h^3$$

$$M_{27} = C_{60}h, \quad M_{28} = -C_{70}h$$

$$M_{29} = 2C_{70}h + 2[C_{69}\lambda + C_{73}]h^2 - [C_{68}\lambda(\lambda-1) + C_{72}\lambda + C_{75}]h^3$$

$$M_{30} = -4[C_{69}\lambda + C_{73}]h^2 + 2[C_{67}\lambda(\lambda-1)(\lambda-2) + C_{71}\lambda(\lambda-1) + C_{74}\lambda + C_{76}]h^4$$

$$M_{31} = -2C_{70}h + 2[C_{69}\lambda + C_{73}]h^2 + [C_{68}\lambda(\lambda-1) + C_{72}\lambda + C_{75}]h^3$$

$$M_{32} = C_{70}h, \quad M_{33} = 2C_{47} - [C_{46}(\lambda+1) + C_{51}]h$$

$$M_{34} = -8C_{47} + 2[C_{46}(\lambda+1) + C_{51}]h + 2[C_{45}(\lambda+1)\lambda + C_{50}(\lambda+1) + C_{54}]h^2 - [C_{44}(\lambda+1)\lambda(\lambda-1) + C_{49}(\lambda+1)\lambda + C_{53}(\lambda+1) + C_{56}]h^3$$

$$M_{35} = 12C_{47} - 4[C_{45}(\lambda+1)\lambda + C_{50}(\lambda+1) + C_{54}]h^2 + 2[C_{43}(\lambda+1)\lambda(\lambda-1)(\lambda-2) + C_{48}(\lambda+1)\lambda(\lambda-1) + C_{52}(\lambda+1)\lambda + C_{55}(\lambda+1)]h^4$$

$$M_{36} = -8C_{47} - 2[C_{46}(\lambda+1) + C_{51}]h + 2[C_{45}(\lambda+1)\lambda + C_{50}(\lambda+1) + C_{54}]h^2 + [C_{44}(\lambda+1)\lambda(\lambda-1) + C_{49}(\lambda+1)\lambda + C_{53}(\lambda+1) + C_{56}]h^3$$

$$M_{37} = 2C_{47} + [C_{46}(\lambda+1) + C_{51}]h.$$

Note that $M_j = M_j(\theta_k, \lambda)$ for $j = 1, 2, \ldots, 37$ and $k = 1, 2, \ldots, n$. Equations (46)–(48) constitute a system of $3n$ equations in $3n+12$ unknowns: $f_{-1}, f_0, f_1, \ldots, f_n, f_{n+1}, f_{n+2}$; $g_{-1}, g_0, g_1, \ldots, g_n, g_{n+1}, g_{n+2}$; and $p_{-1}, p_0, p_1, \ldots, p_n, p_{n+1}, p_{n+2}$.

Boundary conditions for simply-supported edges at $\theta = 0, \alpha$ are given by, using (31)–(34)

$$p_0 = 0, \quad p_{n+1} = 0 \qquad (49),(50)$$

$$\begin{aligned}
&-(\hat{B}_{26}h)f_{-1} + [2(\hat{B}_{21}\lambda + \hat{B}_{22})h^2]f_0 + (\hat{B}_{26}h)f_1 \\
&-(\hat{B}_{22}h)g_{-1} + [2\hat{B}_{26}(\lambda-1)h^2]g_0 + (\hat{B}_{22}h)g_1 \\
&-(2\hat{D}_{22} - 2\hat{D}_{26}\lambda h)p_{-1} - (2\hat{D}_{22} + 2\hat{D}_{26}\lambda h)p_1 = 0, \qquad (51)\\
&-(\hat{B}_{26}h)f_n + [2(\hat{B}_{21}\lambda + \hat{B}_{22})h^2]f_{n+1} + (\hat{B}_{26}h)f_{n+2} \\
&-(\hat{B}_{22}h)g_n + [2\hat{B}_{26}(\lambda-1)h^2]g_{n+1} + (\hat{B}_{22}h)g_{n+2} \\
&-(2\hat{D}_{22} - 2\hat{D}_{26}\lambda h)p_n - (2\hat{D}_{22} + 2\hat{D}_{26}\lambda h)p_{n+2} = 0, \qquad (52)\\
&-(\hat{A}_{66}h)f_{-1} + [2(\hat{A}_{61}\lambda + \hat{A}_{62})h^2]f_0 + (\hat{A}_{66}h)f_1 \\
&-(\hat{A}_{62}h)g_{-1} + [2\hat{A}_{66}(\lambda-1)h^2]g_0 + (\hat{A}_{62}h)g_1 \\
&-(2\hat{B}_{62} - 2\hat{B}_{66}\lambda h)p_{-1} - (2\hat{B}_{62} + 2\hat{B}_{66}\lambda h)p_1 = 0, \qquad (53)\\
&-(\hat{A}_{66}h)f_n + [2(\hat{A}_{61}\lambda + \hat{A}_{62})h^2]f_{n+1} + (\hat{A}_{66}h)f_{n+2} \\
&-(\hat{A}_{62}h)g_n + [2\hat{A}_{66}(\lambda-1)h^2]g_{n+1} + (\hat{A}_{62}h)g_{n+2} \\
&-(2\hat{B}_{62} - 2\hat{B}_{66}\lambda h)p_n - (2\hat{B}_{62} + 2\hat{B}_{66}\lambda h)p_{n+2} = 0, \qquad (54)\\
&-(\hat{A}_{26}h)f_{-1} + [2(\hat{A}_{21}(\lambda-1)h^2]f_0 + (\hat{A}_{22}h)f_1 \\
&-(\hat{A}_{22}h)g_{-1} + [2\hat{A}_{26}(\lambda-1)h^2]g_0 + (\hat{A}_{22}h)g_1 \\
&-(2\hat{B}_{22} - 2\hat{B}_{26}\lambda h)p_{-1} - (2\hat{B}_{22} + 2\hat{B}_{26}\lambda h)p_1 = 0, \qquad (55)
\end{aligned}$$

$$-(\hat{A}_{26}h)f_n + [2(\hat{A}_{21}\lambda + \hat{A}_{22})h^2]f_{n+1} + (\hat{A}_{26}h)f_{n+2}$$
$$-(\hat{A}_{22}h)g_n + [2\hat{A}_{26}(\lambda - 1)h^2]g_{n+1} + (\hat{A}_{22}h)g_{n+2}$$
$$-(2\hat{B}_{22} - 2\hat{B}_{26}\lambda h)p_n - (2\hat{B}_{22} + 2\hat{B}_{26}\lambda h)p_{n+2} = 0, \tag{56}$$

where (49) and (50) were used in writing (51)–(56). Noting (49) and (50), eqns (46)–(48) and (51)–(56) now constitute a system of $3n + 6$ equations in $3n + 10$ unknowns $f_{-1}, f_0, f_1, \ldots, f_n, f_{n+1}, f_{n+2}$; $g_{-1}, g_0, g_1, \ldots, g_n, g_{n+1}, g_{n+2}$; and $p_{-1}, p_1, \ldots, p_n, p_{n+2}$ for a given n. In contrast with the symmetric problem, there are more unknowns than equations.† Presumably there are several ways to reduce the number of unknowns from $3n + 10$ to $3n + 6$. The procedure adopted here only involves changes in the finite difference approximations used for some of the derivatives of the functions $F_1(\theta)$ and $F_2(\theta)$ at some of the mesh points.

In the differential equation (28) at the first "inside" point θ_1, the previously used central difference approximation for the third derivatives: $F_1'''(\theta)$ and $F_2'''(\theta)$ is replaced by the forward difference approximation for the first derivative of the second derivatives. The second derivatives themselves are approximated by the central difference scheme.

Similarly, at the last "inside" point θ_n, we use the backward difference approximation for the first derivative of the second derivatives.

No other changes are made in the finite difference treatment of any of the differential equations (26)–(28). In particular, this means that the central difference scheme is used to approximate all derivatives of $F_3(\theta)$ at all the inside points θ_k; $k = 1, 2, \ldots, n$.

In the boundary conditions (which are only written at the "end" points: θ_0 $(=0)$ and $\theta_{n+1}(=\alpha)$), the first derivatives: $F_1'(\theta)$ and $F_2'(\theta)$ are approximated by forward differences at θ_0, and by backward differences at θ_{n+1}. No other changes are made in the finite difference treatment of the boundary conditions, so that all derivatives of $F_3(\theta)$ are approximated by central differences. The above modifications eliminate f_{-1}, f_{n+2}, g_{-1}, and g_{n+2} as unknowns, with the result that modifications occur in the finite difference versions of some of the differential equations and boundary conditions.

In (28) the equations at θ_1 and θ_n are now replaced by

$$(\bar{M}_{24})f_0 + (\bar{M}_{25})f_1 + (\bar{M}_{26})f_2 + (\bar{M}_{27})f_3 + (\bar{M}_{29})g_0 + (\bar{M}_{30})g_1 + (\bar{M}_{31})g_2 + (\bar{M}_{32})g_3$$
$$+ (\bar{M}_{33})p_{-1} + (M_{35})p_1 + (M_{36})p_2 + (M_{37})p_3 = 0, \tag{57}$$

and

$$(M^*_{23})f_{n-2} + (M^*_{24})f_{n-1} + (M^*_{25})f_n + (M^*_{26})f_{n+1}$$
$$+ (M^*_{28})g_{n-2} + (M^*_{29})g_{n-1} + (M^*_{30})g_n + (M^*_{31})g_{n+1}$$
$$+ (M_{33})p_{n-2} + (M_{34})p_{n-1} + (M_{35})p_n + (M_{37})p_{n+2} = 0, \tag{58}$$

where

$$\bar{M}_{24} = M_{24} - 4C_{60}h, \quad \bar{M}_{25} = M_{25} + 6C_{60}h, \quad \bar{M}_{26} = M_{26} - 4C_{60}h$$
$$\bar{M}_{27} = 2M_{27}, \quad \bar{M}_{29} = M_{29} - 4C_{70}h, \quad \bar{M}_{30} = M_{30} + 6C_{70}h$$
$$\bar{M}_{31} = M_{31} - 4C_{70}h, \quad \bar{M}_{32} = 2M_{32},$$

and

$$M^*_{23} = 2M_{23}, \quad M^*_{24} = M_{24} + 4C_{60}h, \quad M^*_{25} = M_{25} - 6C_{60}h$$
$$M^*_{26} = M_{26} + 4C_{60}h, \quad M^*_{28} = M_{28}, \quad M^*_{29} = M_{29} + 4C_{70}h$$
$$M^*_{30} = M_{30} - 6C_{70}h, \quad M^*_{31} = M_{31} + 4C_{70}h.$$

† This situation will also arise in the symmetric problem if a free-edge boundary condition is prescribed on either or both of: $\theta = 0$ and $\theta = \alpha$.

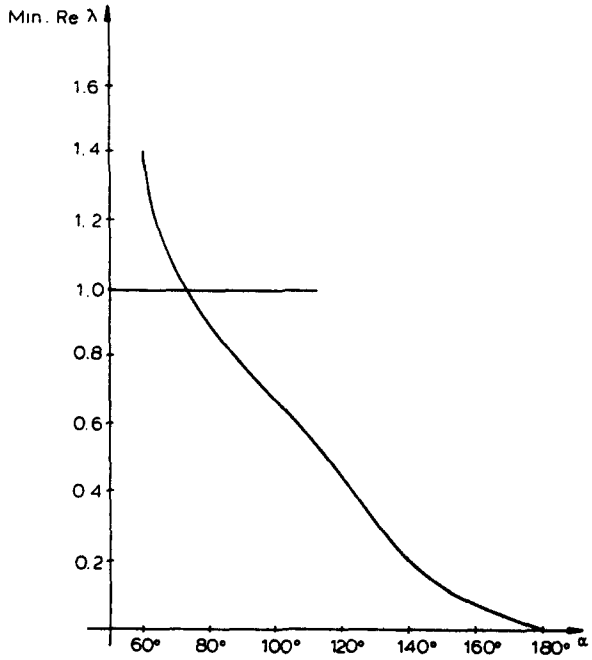

Fig. 4. Antisymmetric problem ($\theta_0 = 15°$): minimum Re λ vs sector angle α curve.

Boundary conditions (51)–(56) now read

$$[(\hat{B}_{21}\lambda + \hat{B}_{22})h^2 - \hat{B}_{26}h]f_0 + (\hat{B}_{26}h)f_1 + [\hat{B}_{26}(\lambda - 1)h^2 - \hat{B}_{22}h]g_0 + (\hat{B}_{22}h)g_1 - (\hat{D}_{22} - \hat{D}_{26}\lambda h)p_{-1} - (\hat{D}_{22} + \hat{D}_{26}\lambda h)p_1 = 0. \tag{59}$$

$$-(\hat{B}_{26}h)f_n + [\hat{B}_{26}h + (\hat{B}_{21}\lambda + \hat{B}_{22})h^2]f_{n+1} - (\hat{B}_{22}h)g_n + [\hat{B}_{22}h + \hat{B}_{26}(\lambda - 1)h^2]g_{n+1} - (\hat{D}_{22} - \hat{D}_{26}\lambda h)p_n - (\hat{D}_{22} + \hat{D}_{26}\lambda h)p_{n+2} = 0, \tag{60}$$

$$[(\hat{A}_{61}\lambda + \hat{A}_{62})h^2 - \hat{A}_{66}h]f_0 + (\hat{A}_{66}h)f_1 + [\hat{A}_{66}(\lambda - 1)h^2 - (\hat{A}_{62}h)]g_0 + (\hat{A}_{62}h)g_1 - (\hat{B}_{62} - \hat{B}_{66}\lambda h)p_{-1} - (\hat{B}_{62} + \hat{B}_{66}\lambda h)p_1 = 0, \tag{61}$$

$$-(\hat{A}_{66}h)f_n + [\hat{A}_{66}h + (\hat{A}_{61}\lambda + \hat{A}_{62})h^2]f_{n+1} - (\hat{A}_{62}h)g_n + [\hat{A}_{62}h + \hat{A}_{66}(\lambda - 1)h^2]g_{n+1} - (\hat{B}_{62} - \hat{B}_{66}\lambda h)p_n - (\hat{B}_{62} + \hat{B}_{66}\lambda h)p_{n+2} = 0, \tag{62}$$

$$[(\hat{A}_{21}\lambda + \hat{A}_{22})h^2 - (\hat{A}_{26}h)]f_0 + (\hat{A}_{26}h)f_1 + [\hat{A}_{26}(\lambda - 1)h^2 - (\hat{A}_{22}h)]g_0 + (\hat{A}_{22}h)g_1 - (\hat{B}_{22} - \hat{B}_{26}\lambda h)p_{-1} - (\hat{B}_{22} + \hat{B}_{26}\lambda h)p_1 = 0, \tag{63}$$

and

$$-(\hat{A}_{26}h)f_n + [\hat{A}_{26}h + (\hat{A}_{21}\lambda + \hat{A}_{22})h^2]f_{n+1} - (\hat{A}_{22}h)g_n + [\hat{A}_{22}h + \hat{A}_{26}(\lambda - 1)h^2]g_{n+1} - (\hat{B}_{22} - \hat{B}_{26}\lambda h)p_n - (\hat{B}_{22} + \hat{B}_{26}\lambda h)p_{n+2} = 0. \tag{64}$$

Equations (46), (47), (56), (48) (with $k = 2, 3, \ldots, n-1$), (57)–(64) now constitute a system of $3n + 6$ equations in $3n + 6$ unknowns $f_0, f_1, \ldots, f_n, f_{n+1}$; $g_0, g_1, \ldots, g_n, g_{n+1}$; and $p_{-1}, p_1, \ldots, p_n, p_{n+2}$. Hence

$$[A(\lambda)]\begin{Bmatrix}\{f\}\\ \{g\}\\ \{p\}\end{Bmatrix} = 0. \tag{65}$$

For non-trivial solutions for $\{f\}$, $\{g\}$ and $\{p\}$

$$\det([A(\lambda)]) = 0. \tag{66}$$

Equation (66) determines the singularity parameter λ.

For a given n, the number of matrix elements in the antisymmetric case is nine times that of the symmetric case. Obtaining λ in the antisymmetric case was found to be about five times as expensive as the symmetric case.

Using the methods described above with $n = 18$,† results were obtained for a graphite/epoxy T-300/5208 angle-ply wedge of thickenss 2 mm, consisting of eight layers of equal thickness. Shown in Fig. 4 are the values of min Re λ for various sector angles α and lay-up [15°/−15°]. Here, singular stresses start to appear at sector angle 73°. This is a greater sector angle compared to the corresponding symmetric case.

Fiber angles other than the [15°/−15°] case were not considered in view of expense. However, we do believe that the feasibility of the method has been demonstrated.

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†The values $n = 12$ and $n = 18$ were tested in computations for the largest sector angle α. No significant deviation was found between the two.